

RECEIVED: August 16, 2015

REVISED: January 4, 2016

ACCEPTED: January 31, 2016

PUBLISHED: February 22, 2016

4d/3d reduction of s-confining theories: the role of the “exotic” D instantons

Antonio Amariti

LPTENS,

24 Rue Lhomond, Paris 75005, France

Physics Department,

CCNY 160 Convent Avenue, New York, NY 10031, U.S.A.

E-mail: aamariti@ccny.cuny.edu

ABSTRACT: The reduction of 4d Seiberg duality to 3d by compactification on a circle is possible if finite size effects are considered. These effects boil down to the contribution of KK monopole operators acting as instantons in 3d, and they are crucial to preserve the 4d duality in 3d. This mechanism has been reproduced in string theory by T-duality on the type IIA brane setup. In some cases the 4d dual “magnetic” theories are IR confined descriptions of the UV gauge theories. In these cases the monopoles are absent in the IR dynamics and the mechanism of reduction of the 4d duality has to be modified. In this paper we investigate such modification in the brane setup. The main observation behind our analysis is that in the 4d case the superpotential of the confined theories can be obtained also from the “exotic” contribution of a D0 brane, a stringy instanton. When considering these configurations we reproduce the field theory results in the brane setup. We study both the unitary and the symplectic case. As a further check we provide the interpretation of the mechanism in terms of localization.

KEYWORDS: Supersymmetry and Duality, Confinement, Intersecting branes models

ARXIV EPRINT: [1507.05623](https://arxiv.org/abs/1507.05623)

Contents

1	Introduction	1
2	s-confinement and exotic instantons	3
3	Reduction to 3d of $SU(N_c)$ SQCD with $N_c + 1$ flavors	5
4	The symplectic case	11
5	Conclusions	14

1 Introduction

A general procedure for reducing 4d dualities to 3d has been developed in [1, 2] (see also [3] for an earlier discussion). It is based on the observation that a straight compactification of dual theories on a circle generally spoils the original 4d duality. This is because when dimensionally reducing the theories the scale of validity of the 3d duality is lower than the effective scale at which the new theories have to be considered. Alternatively one can think that in this process extra symmetries, anomalous in 4d, are generated in 3d. These symmetries mix with the R-current, leading the 4d fixed points, identified by the duality, to different fixed points in the 3d flow.

This problem can be avoided if the finite size effects of the circle are considered. In this way there is an effective 3d duality at the scale set by the circle. At such scale there are both BPS and KK monopoles, acting as instantons in the 3d setup and generating non perturbative superpotentials. These superpotentials break the 4d anomalous symmetries discussed above. The theories on the circle can be treated as effective 3d dualities if this finite size effect is considered. The more conventional 3d dual pairs of [4] are obtained by a real mass flow.

This procedure is general and can be applied to any 4d duality. It has been shown that, if the theories have a IIA brane realization, the mechanism can be reproduced by string theory [5, 6]. This is based on T-duality and the effects of the monopoles are reproduced by the action euclidean D-branes.

There is another possible IR behavior of UV free theories in 4d: confinement. In this case the low energy dynamics is not described by a dual gauge theory but in terms of mesons and baryons. In this case the reduction to 3d is more complicated and the prescription of [1, 2] requires some modification. Before describing the reduction in the confining phase we discuss some 4d aspects of these theories that will be useful in the following.

In supersymmetry some confining theories correspond to limiting cases of electric/magnetic Seiberg dualities [7]. The simplest example is $\mathcal{N} = 1$ SQCD with N colors and $N + 1$

flavors. This theory is s-confining¹ in the IR [9], and in this regime it is described by the mesonic and the baryonic degrees of freedom, interacting through a superpotential. This superpotential corresponds to the classical constraint on the moduli space. Equivalently this low energy description has been obtained by adding a massive flavor in the gauge theory and by studying the large mass regime [7]. In this case the theory has originally $N + 2$ flavors and the strongly coupled phase can be described by a dual IR free $SU(2)$ gauge theory. Integrating out the massive flavor in the electric theory corresponds to a total higgsing of the $SU(2)$ dual gauge group. In this case the superpotential is reproduced by a scale matching relation on the instantonic contribution of the totally broken $SU(2)$.

Gauge instantons have a counterpart in string theory: they are related to Dp branes placed on stacks of $D(p+4)$ branes (see [10, 11] and references therein for review). The stack of $D(p+4)$ branes corresponds to the non-abelian gauge theory and the Dp branes reproduce the effect of the gauge instantons.

The gauge instanton effect discussed above can be captured in a different way in string theory, without requiring to UV complete the confining phase to an $SU(2)$ gauge theory. This effect can be observed by a IIA brane engineering, by considering a single D4 brane extended between two non parallel NS branes, with in addition some D6 flavor branes. In 4d the abelian gauge factor associated to this D4 brane decouples in the IR. Unexpectedly, also in this case, the D instanton, an euclidean D0 brane, reproduces the superpotential effect of the gauge instanton of the broken $SU(2)$ [12–14]. For this reason this D instanton has been called “exotic” in the literature. Observe that the size of the instantonic correction is different in the two regimes, the stringy and the field theory one, signaling that the two descriptions are accurate at different scales [13].

In this paper we study the fate of the stringy instantons in s-confining theories compactified on a circle and reduced to 3d. We perform the reduction separately in the confining and in the confined phase² both in the field theory regime and in the string theory regime. In the field theory regime we follow the procedure of [1, 2] for reducing the UV confining phase. The IR confined regime corresponds to a set of mesonic and baryonic operators, interacting through a superpotential, without gauge degrees of freedom. In this case there is a different prescription for the reduction to 3d [15, 16] and the effective IR confined theory on the circle has the same field content and superpotential of the 4d parent. The classical constraint enforced in the IR theory to match the moduli space of the UV theory corresponds to the 3d instantonic effect discussed above.

We show how this result is reproduced in string theory. In such a regime we follow the arguments of [5, 6] for reducing the confining gauge theory on the circle. In the confined case we observe that the instantonic contribution is reproduced by the T-dual version of the euclidean D0 instanton. This corresponds to a D1 euclidean brane, i.e. a monopole

¹Using the original terminology of [8] we refer to an s-confining theory as a theory with smooth confinement without chiral symmetry breaking and with a non-vanishing confining superpotential.

²Observe that in 4d, in the limiting case of $SU(N)$ Seiberg duality, the electric theory becomes a confining phase while the magnetic theory is identified with the confined phase. In the 3d case, where the theories are conformal, we have a duality. For this reason, with a slight abuse of notation, in the 3d case, we refer to the two dual phases as to the electric and the magnetic theory.

acting as an instanton in 3d. Thanks to this observation we reproduce the results obtained in field theory with the prescription of [15, 16]. We study the reduction of 4d confining theories both with unitary and symplectic gauge groups. Furthermore we study how this mechanism is captured by localization of the path integral on $S^3 \times S^1$.

In section 2 we review the 4d s-confining $SU(N)$ SQCD and the relation with the stringy instanton. In section 3 we study the reduction of this theory to 3d in field theory and in string theory. We study the role of the stringy instanton in the brane engineering of this theory. In the 3d limit, by gauging the baryonic symmetry, we arrive at the $U(N)$ case with N flavors, and reproduce the limiting case of Aharony duality [4]. We conclude this section with the reduction of the localized path integral on $S^3 \times S^1$ to the partition function on the squashed S^3 . It confirms the validity of the procedure. In section 4 we repeat the analysis for the symplectic case. In section 5 we conclude.

2 s-confinement and exotic instantons

In this section we review the exotic effects of stringy instantons in 4d $\mathcal{N} = 1$ supersymmetric gauge theories. This instanton configuration has been extensively studied in cascading quiver gauge theories,³ associated to the CY singularity probed by a stack of D3 brane in AdS/CFT [12, 19–25]. The RG cascading quivers are obtained by the addition of fractional D3 branes. There are cases where some of the nodes have rank $N = 1$, i.e. a single D3 is left at the end of the duality cascade on such nodes. With an abuse of notation we denote these nodes as $SU(1)$ nodes, having in mind the decoupling of the $U(1)$ factors in the IR. Even if there are no gauge dynamical degrees of freedom from the singularities associated to these nodes, the latter play a role in the dynamics if euclidean rigid $D(-1)$ instantons are wrapped on them. The instantons generate a non perturbative dynamics that modifies the effective superpotential. This is obtained by considering the bosonic and fermionic zero modes in the ADHM construction and the relative action and constraints. Many of the zero modes are lifted, except for two fermionic modes that correspond to two fields, α and β , also called Ganor strings [26], connecting the $SU(1)$ node to the rest of the quiver. These modes are lifted by an interaction $\alpha M \beta$, giving origin to the superpotential

$$W = \int d\alpha d\beta e^{\alpha_a M_{ab} \beta_b} \simeq \det M \quad (2.1)$$

Observe that the fermionic zero modes and the generalized meson M have an index structure inherited from the quiver.

This is reminiscent of the mesonic superpotential appearing in the confined phase of $SU(N)$ SQCD with $N + 1$ flavors [12–14]. As discussed in the introduction this theory confines in the IR and it can be obtained as a limiting case of Seiberg duality. The low energy theory in this case has superpotential of the form

$$W = b M \tilde{b} + \det M \quad (2.2)$$

³Similar ideas have been discussed in the context of matrix model [17]. The relation with the stringy instanton has been shown in [18].

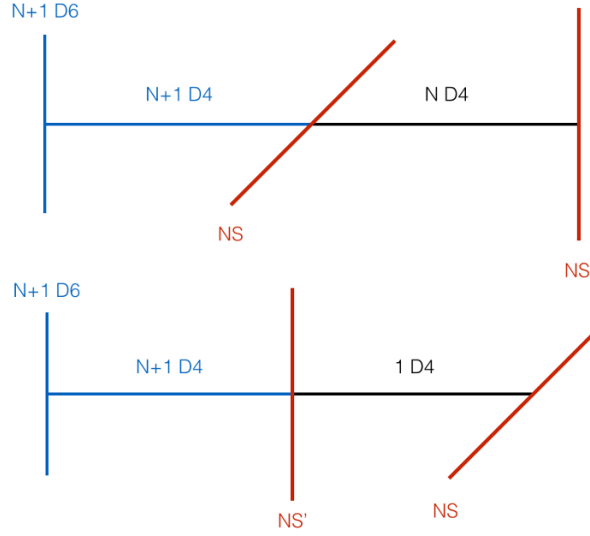


Figure 1. The top figure represents the IIA brane setup describing $SU(N)$ SQCD with $N + 1$ flavors, the bottom figure represents the theory after an HW transition.

The second term in (2.2) has been interpreted as a gauge instanton in field theory, by UV completing the s-confining phase to $SU(N)$ SQCD with $N + 2$ flavors. In this case there is a Seiberg dual description in terms of an $SU(2)$ gauge theory with $N + 2$ dual flavors. If a mass term $W = mQ_{N+2}^\alpha \tilde{Q}_\alpha^{N+2}$ is added the IR theory has $N + 1$ light flavors. In the magnetic theory the mass term enforces the total higgsing of the $SU(2)$ gauge group, thanks to the F term of the meson M_{N+2}^{N+2} . In the higgsed phase the dual quarks are identified with the baryons of the electric theory with $N + 1$ flavors. The gauge instanton associated to the broken of $SU(2)$ generates a contribution proportional to $\det M$ in the superpotential. This construction reproduces the superpotential (2.2).

As discussed above this contribution can be obtained also from an instantonic calculation by engineering the gauge theory in an Hanany-Witten (HW) [27] setup. The electric SQCD theory is represented as the low energy limit of a stack of N D4 branes in type IIA string theory. Here we review the construction of this theory.

The 4d gauge theory is described by a stack of N D4 branes suspended between an NS and an NS' brane. When $N + 1$ D6 are considered this setup describes $SU(N)$ SQCD with $N + 1$ flavors. The branes are extended in the ten dimensional space-time as follows:

	0	1	2	3	4	5	6	7	8	9
D4	×	×	×	×			×			
D6	×	×	×	×				×	×	×
NS	×	×	×	×	×	×				
NS'	×	×	×	×					×	×
D0							×			

The brane setup is shown in figure 1. The confined theory is obtained by exchanging the NS and the NS' branes. A single D4 brane is stretched between the NS and the NS' brane

after the transition. This corresponds to a $U(1)$ gauge theory. In 4d the vector multiplet of an abelian gauge theory decouples in the IR. We are in presence of the $SU(1)$ factor discussed above.

The light degrees of freedom in this setup are the baryons, oriented strings connecting the $N + 1$ flavor D4 branes with the D4 brane stretched between the NS and the NS' brane, and the meson, open strings with both the endpoints on the $N + 1$ flavor D4 branes. The meson is dynamical because it is associated to the freedom in moving the $N + 1$ flavor D4 branes in the directions $(8, 9)$. This motion corresponds to an effective mass for the baryons and it can be represented as a superpotential term

$$W = M b \tilde{b} \quad (2.3)$$

In other words the dynamical mass term of the baryon corresponds to the vev of the meson M in the $(8, 9)$ directions. The description of the confined theory in terms of D branes requires also a term proportional to $\det M$ in (2.3). This term is obtained from the action of the exotic instanton, i.e. an euclidean D0 brane on the D4 brane stretched between the NS and the NS' branes.

3 Reduction to 3d of $SU(N_c)$ SQCD with $N_c + 1$ flavors

In this section we reduce the 4d theories discussed above to 3d, in the field theory and in the string theory regime. We also study how this reduction reflects in the supersymmetric localization of the partition function on the warped product of S^3 and S^1 to the partition function on the squashed three sphere, S_b^3 .

Dimensional reduction in the field theory regime. When reducing a 4d duality to 3d a straight compactification may be too naive, and the 3d pairs obtained are not necessarily dual. Indeed 4d anomalous abelian symmetries may arise in 3d and mix with the current J_μ^R . This mixing can induce different RG flows in the two phases, spoiling the original 4d duality [1].

There is a general procedure to reduce a 4d dual pair to a 3d one. It consists of considering the 4d theories on $\mathbb{R}^3 \times S^1$ as effective 3d theories. The finite size of the circle translates into an effective superpotential [1]. This effective interaction prevents the generation of 4d anomalous symmetries in 3d. This procedure generates 3d dual effective theories. More conventional dual pairs are recovered by shrinking the circle S^1 . In some cases this limit requires an additional real mass deformation to be well defined.

This general procedure is valid when both the phases of the duality correspond to a gauge theory. In these cases the extra non perturbative effects can be thought as the fractionalization of a 4d instanton into 3d monopoles [28]. These monopoles, acting as instantons in 3d, are of two types. There are BPS monopoles, that survive the compactification and KK monopoles, that encode the information of circle at finite radius [27, 29–31].

In the limiting cases considered here, where instead of a dual theory the IR physics is described by a confined phase, there are no instanton contributions in the low energy theory, because the gauge group vanishes. Anyway there is a contribution to the superpotential

of non-perturbative origin. This effect corresponds to the instanton of the totally broken $SU(2)$ as discussed above. One can study the $SU(2)$ dual gauge theory on $\mathbb{R}^3 \times S^1$, consider the effect of the KK monopole and completely break the dual gauge symmetry as done in the 4d case. A different strategy has been proposed in [15]: when reducing a confined phase on the circle the effective theory is formally identical to the 4d parent. We will adopt this strategy in the rest of the discussion.

First we reduce the electric gauge theory, $SU(N)$ SQCD with $N+1$ flavors, on $\mathbb{R}^3 \times S^1$. Here there is a KK monopole contribution, through the effective superpotential

$$W = \eta Y \quad \text{where} \quad Y = e^{(\sigma_1 - \sigma_{N_c})/g_3^2 + i(\varphi_1 - \varphi_N)}. \quad (3.1)$$

We will refer to this superpotential as the ηY superpotential in the rest of the paper. The fields σ are the real scalars in the vector multiplet and the fields φ correspond to the dual photons, in the Coulomb branch. These two fields organize in a chiral multiplet $\Sigma = \sigma/g_3^2 + i\varphi$ that parameterizes the Coulomb branch of the moduli space. The operator $Y = e^\Sigma$ corresponds to a monopole operator in the high energy description. The 4d gauge coupling g_4 reduces to the 3d one by the relation $g_4^2 = 2\pi r g_3^2$. The extra superpotential (3.1) is associated to the holomorphic scale of the 4d theory by $\Lambda^b = \eta = e^{\frac{4\pi}{r g_3^2}}$. The confined case on $\mathbb{R}^3 \times S^1$ corresponds to the set of mesons and baryons discussed above. The superpotential of this theory is again (2.3).

The pure 3d duality is obtained from the duality on the circle by perturbing the electric theory with a real mass deformation. We assign one large real mass to a pair of fundamental and anti-fundamental fields and reduce the flavor from $N+1$ to N . This is done by weakly gauging a combination of generators of the baryonic symmetry and of the non abelian symmetry as discussed in the appendix of [15]. We choose a combination that assigns the opposite sign to the real masses of the two fields. In the large mass limit the electric theory becomes $SU(N)$ SQCD with N flavors. The masses have opposite sign and the flow does not generate any CS term. This deformation induces non vanishing real masses for some components of the mesons and of the baryons in the dual theory. These masses are assigned consistently with the global symmetry structure. Concretely, in the dual theory, if we split the mesons and the baryons as

$$M = \begin{pmatrix} M_i^i & M_i^{N+1} \\ M_{N+1}^i & M_{N+1}^{N+1} \end{pmatrix} \quad B = (B_i \ B_{N+1}) \quad \tilde{B} = \begin{pmatrix} \tilde{B}^i \\ \tilde{B}^{N+1} \end{pmatrix} \quad (3.2)$$

the massless components are M_i^i , M_{N+1}^{N+1} , B_{N+1} and \tilde{B}^{N+1} . The superpotential for the massless fields is

$$W = M_{N+1}^{N+1} (B_{N+1} \tilde{B}^{N+1} + \det M_i^i) \quad (3.3)$$

The singlet M_{N+1}^{N+1} has the same global charges of the electric monopole Y defined in (3.1). By further identifying M_{N+1}^{N+1} with Y we obtain the 3d duality corresponding to the limiting case of $SU(N)$ Aharony duality [4].

We can also gauge the $U(1)_B$ baryonic symmetry. The electric theory becomes $U(N)$ SQCD with N flavors. The dual theory becomes SQED with one flavor and it is mirror dual

to the \mathcal{XYZ} model [32]. Here we associate the field \mathcal{X} to the gauge invariant combination $\tilde{B}^{N+1}B_{N+1}$, while the other two chiral fields can be denoted as $\mathcal{Y} = v_+$ and $\mathcal{Z} = v_-$. The superpotential of the mirror dual theory becomes

$$W = Y\mathcal{X} + Y \det M_i^i + v_+ v_- \mathcal{X} \quad (3.4)$$

By integrating out the massive fields we obtain the relation $Y = v_+ v_-$. Eventually the superpotential of the dual theory is

$$W = v_+ v_- \det M_i^i \quad (3.5)$$

and it corresponds to the superpotential of the limiting case of $U(N)$ Aharony duality [4].

Brane interpretation. Here we provide a brane interpretation of the reduction discussed above. A general procedure for reducing 4d dualities engineered in type IIA string theory to 3d dualities in IIB setups has been developed in [5] for unitary theories with fundamental flavor, and extended in [6] to more general gauge and field contents. The procedure is based on compactification and T-duality. The KK monopole effects are captured by D1-branes or, by S-duality, by F-strings. Reducing to pure 3d pairs requires a double scaling limit on the radius and the real masses, associated to the position of some flavor brane on the circle.

Here we consider the 4d brane setup of figure 1, and compactify the theory on x_3 . By T-duality along this direction the IIA system becomes a IIB system and describes the effective 4d field theory on $R^3 \times S_r^1$, where r is the radius of the circle. The NS and NS' branes are left invariant by T-duality while the D4 and the D6 branes become D3 and D5 branes respectively. The D0 branes become D1 branes. At large T-dual radius α'/r this brane setup describes an effective 3d theory.

When considering a 3d duality at brane level we must also gauge the $U(1)_B$ symmetry [5]. At brane level we associate this symmetry to the relative position of the center of mass of the stack of the gauge D3 branes with respect to the position of the center of mass of the flavor D3 branes. By fixing the position of the center of mass of one stack and allowing the motion of the other one can consider the $U(1)_B$ symmetry as gauged or not.

In the $U(N)$ theory on the circle the effect of the monopoles is encoded in the D1 branes stretched between the D3 and the NS branes along the directions x_3 and x_6 . In the 3d decompactified case these effects, corresponding to the repulsive interactions between the parallel D3 branes, are associated to the BPS monopole superpotential, $W \simeq \sum Y_i^{-1}$. In the compact case there is an additional contribution, corresponding to the KK monopole contribution [29, 30].

The analysis of the reduction of the confining phase differs from the one of [5]. Indeed in this case the ηY superpotential cannot arise, because of the absence of the dual gauge group. In the brane picture there is still a D3 brane and we can consider the effect of a D1 brane wrapping the x_3 direction. This is the effect of the 4d stringy instanton once the theory is reduced on the circle. In 4d it gave origin to the extra term $\det M$ in

the superpotential. Here, in the 3d description, this effect is captured by the T-dual D1 brane.

We can further flow to a pure 3d duality by a real mass flow. In the electric case we move a D5 brane at $x_3 = \pi r$ on the circle. By fixing the D3 branes at the position $x_3 = 0$ on the T-dual circle we locate a D5 at the position $x_3 = \pi\alpha'/r$, defined as the mirror point in [6].

By considering the limit $r \rightarrow 0$ the sector at this mirror point can be further decoupled and we are left with 3d $U(N)$ SQCD with N flavors. In the dual theory the motion of a D5 at the mirror point generates a D3 brane by the HW effect. In this case there are no D3 branes left in the gauge sector at $x_3 = 0$. The baryons at $x_3 = 0$ are massive and there is an $N \times N$ meson M left. At small r an S-duality produces an \mathcal{XYZ} -like model at the mirror point. This is not the usual \mathcal{XYZ} model because there are N D3 stretched between a D5 parallel to an NS' brane. This signals the presence of a mass term for one of the singlets, the freedom to move the D3 branes between the D5 and the parallel NS'. The superpotential is $W = \mathcal{XY} + \mathcal{XZ}$, where Y parameterizes the motion of the flavor D3 brane placed at the mirror point.

In the small r limit this field Y interacts with the mesons M_{11} at $x_3 = 0$ through the superpotential generated by the D1 brane. This interaction corresponds indeed to moving one of the D5 brane at $x_3 = \pi\alpha'/r$. The original term $\det M$ becomes in this case $Y \det M_1^1$. Putting everything together the superpotential becomes $W = \mathcal{YZ} \det M_{11}$, that corresponds to (3.5) if the fields \mathcal{Y} and \mathcal{Z} are identified with the monopole and the anti-monopole of the $U(N)$ theory, v_{\pm} .

The partition function. We conclude the analysis of the reduction of the confining $SU(N)$ SQCD with $N + 1$ flavors and of its confined description by studying how this mechanism is reproduced by localization.

Our starting point is the identity of the supersymmetric partition function computed on the curved manifold $S^3 \times S^1$ by localization of the path integral. As shown in [33] in presence of a conserved R-current a supersymmetric field theory can be put on $S^3 \times S^1$. In such a case the path integral can be computed exactly at one loop. Observe that if the theory is superconformal the supersymmetric partition function on $S^3 \times S^1$ coincides with the superconformal index defined in [34, 35] modulo an overall normalization, corresponding to the supersymmetric Casimir energy [36]. This last term does not play a relevant role in our analysis because it coincides among dual theories.

The identities between the supersymmetric partition functions on $S^3 \times S^1$ can be reduced to identities between 3d partition function [37–40], computed on a squashed three sphere S_b^3 [41] preserving an $U(1)^2$ isometry of the original $SO(4)$ of S^3 . The subscript b represents the squashing parameter.

The identity between the supersymmetric partition functions on $S^3 \times S^1$ for the 4d theories has been studied in [42, 43]. After reducing the 4d relations one obtains the 3d partition function of $SU(N)$ SQCD with $N + 1$ flavor with the extra ηY superpotential.

This partition function can be written as

$$\begin{aligned} \mathcal{Z}_e(\mu + m_B; \nu - m_B) &= \int \prod_{i=1}^N d\sigma_i \prod_{a=1}^{N+1} \Gamma_h(\mu_a + \sigma_i + m_B) \Gamma_h(\nu_a - \sigma_i - m_B) \\ &\times \prod_{1 \leq i < j \leq N} \Gamma_h^{-1}(\pm(\sigma_i - \sigma_j)) \delta\left(\sum_{i=1}^N \sigma_i\right) \end{aligned} \quad (3.6)$$

The functions Γ_h , hyperbolic gamma functions [44], are the one loop-exact determinants of the matter and vector multiplets computed from localization in [41, 45, 46]. The parameters μ_a and ν_a correspond to the real masses associated to the $SU(N+1)_l \times SU(N+1)_r$ flavor symmetry. These masses have an imaginary part corresponding to the R -symmetry, i.e. $\mu_a = m_a + \omega\Delta$ and $\nu_a = \tilde{m}_a + \omega\Delta$, where m_a and \tilde{m}_a are real mass parameters satisfying $\sum m_a = \sum \tilde{m}_a = 0$, Δ is the R -charge and $\omega \equiv i(b+1/b)$. The parameter m_B is associated to the real mass for the baryonic $U(1)_B$ symmetry. The real coordinate σ_i corresponds to the scalar in the vector multiplet, and the constraint $\sum \sigma_i = 0$ is enforced by the δ -function. The real scalar parameterizes the fundamental representation as $(+\sigma_i)$, the anti-fundamental as $(-\sigma_i)$ and the adjoint as $(\sigma_i - \sigma_j)$. We used the definition $\Gamma_h(\pm z) = \Gamma_h(z)\Gamma_h(-z)$. There is a constraint between the real masses, that corresponds to the condition imposed by the presence of the superpotential (3.1). This constraint is

$$\sum_{a=1}^{N+1} \mu_a + \sum_{a=1}^{N+1} \nu_a = 2\omega \quad (3.7)$$

We can also study the effect of the gauging of the baryonic $U(1)_B$ symmetry on the partition function. First we introduce a factor $e^{2\pi i m_B \Lambda N}$, where Λ is an arbitrary real parameter. Then we Fourier transform the δ -function, we shift $\sigma_i \rightarrow \sigma_i - m_B$ and obtain a new δ -function $\delta(\Lambda - \xi)$. After performing the integral over ξ we obtain

$$\mathcal{Z}_e(\mu; \nu; \Lambda) = \frac{1}{N} \int e^{2\pi i \Lambda \sum_i \sigma_i} \prod_{i=1}^N d\sigma_i \prod_{a=1}^{N+1} \Gamma_h(\mu_a + \sigma_i) \Gamma_h(\nu_a - \sigma_i) \prod_{1 \leq i < j \leq N} \Gamma_h^{-1}(\pm(\sigma_i - \sigma_j)) \quad (3.8)$$

When reduced on the circle the partition function of the 4d confined phase corresponds to the product of the contributions of the mesons and of the baryons. The 3d partition function of the dual effective theory is

$$\mathcal{Z}_m = \mathcal{Z}_M \mathcal{Z}_b \mathcal{Z}_{\tilde{b}} = \prod_{a,b}^{N+1} \Gamma_h(\mu_a + \nu_b) \prod_{a=1}^{N+1} \Gamma(\omega + Nm_B - \mu_a) \Gamma_h(\omega - Nm_B - \nu_a) \quad (3.9)$$

The partition function in (3.6) coincides with the one in (3.9) if the parameters satisfy (3.7). As done in the electric case we can further add the extra factor $e^{2\pi i m_B \Lambda N}$ and gauge the $U(1)_B$ symmetry. We obtain

$$\mathcal{Z}_m = \prod_{a,b}^{N+1} \Gamma_h(\mu_a + \nu_b) \int dm_B e^{2\pi i Nm_B \Lambda} \prod_{a=1}^{N+1} \Gamma(\omega + Nm_B - \mu_a) \Gamma_h(\omega - Nm_B - \nu_a) \quad (3.10)$$

The decompactification limit, $r_{S_1} \rightarrow 0$, requires a real mass flow in the field theory analysis. This real mass flow is reproduced if we consider the assignation

$$\mu_a = \begin{cases} m_a + m_A & a = 1, \dots, N \\ m - m_A N + \omega & \end{cases} \quad \nu_a = \begin{cases} \tilde{m}_a + m_A & a = 1, \dots, N \\ -m - m_A N + \omega & \end{cases} \quad (3.11)$$

with the constraint $\sum m_a = \sum \tilde{m}_a = 0$. The flow is reproduced by the limit $m \rightarrow \infty$ on the partition function. On the hyperbolic gamma functions this limit is obtained from the formula [44]

$$\lim_{x \rightarrow \infty} \Gamma_h(x) = e^{i\pi \operatorname{sign}(x)(x-\omega)^2} \quad (3.12)$$

We study this limit on the partition function on both sides of the duality. The partition function of the electric theory becomes (we omit the large m dependence in the following because we checked that it coincides to the one computed in the dual frame)

$$\mathcal{Z}_e = \int \frac{e^{2\pi i \Lambda \operatorname{Tr} \sigma}}{N} \prod_{i=1}^N d\sigma_i \prod_{a=1}^N \Gamma_h(m_a + m_A + \sigma_i) \Gamma_h(\tilde{m}_a + m_A - \sigma_i) \prod_{1 \leq i < j \leq N} \Gamma_h^{-1}(\pm(\sigma_i - \sigma_j)) \quad (3.13)$$

This formula corresponds to the partition function of the $U(N)$ gauge theory with N flavors. In the magnetic case we first rescale m_B as m_B/N and then assign the real masses. In the large M limit we have

$$\mathcal{Z}_m = \Gamma_h(2\omega - 2Nm_A) \prod_{a,b}^N \Gamma_h(m_a + \tilde{m}_b + 2m_A) \int \frac{dm_B}{N} e^{2\pi i m_B \Lambda} \Gamma(\pm m_B + Nm_A) \quad (3.14)$$

The last step consists of using the duality between SQED with one flavor and the \mathcal{XYZ} model. On the partition function this duality is encoded in the relation [47]

$$\int dm_B e^{2\pi i m_B \Lambda} \Gamma(\pm m_B + Nm_A) = \Gamma_h(2Nm_A) \Gamma_h\left(\pm \frac{\Lambda}{2} - Nm_A + \omega\right) \quad (3.15)$$

The product $\Gamma_h(2Nm_A) \Gamma_h(2\omega - 2Nm_A)$ is compatible with a superpotential mass term and it can be further simplified by the relation $\Gamma_h(x) \Gamma_h(2\omega - x) = 1$. We obtain

$$\mathcal{Z}_m = \prod_{a,b}^N \Gamma_h(m_a + \tilde{m}_b + 2m_A) \Gamma_h\left(\pm \frac{\Lambda}{2} - Nm_A + \omega\right) \quad (3.16)$$

that is equivalent to (3.13) and describes the partition function of the dual theory with superpotential (3.5). Observe the role of Λ : it is an FI term in the electric theory, added when gauging the baryonic symmetry. This parameter becomes a real mass parameter in the magnetic theory. This is expected because the FI parameter corresponds to the real mass parameter of the $U(1)_J$ symmetry. This is a topological symmetry that shifts the dual photon and that arises only in the $U(N)$ case. In the dual theory the topological symmetry does not disappear, even if the dual gauge theory is trivial, because there are gauge singlets, the electric monopole operators, carrying a non trivial charge under $U(1)_J$.

4 The symplectic case

In this section we study the dimensional reduction of an s-confining theory with symplectic gauge group. Also in this case the superpotential of the confined phase can be obtained from the action of a stringy instanton in the IR theory. This stringy instanton contributes to the effective Lagrangian of an $SP(0)$ theory (as in the $SU(1)$ case here we use a similar abuse of terminology) and corresponds to an $O(1)$ instanton.

The $O(1)$ instanton in 4d. Here we consider a 4d $\mathcal{N} = 1$ $SP(2N)$ gauge theory⁴ with $2(N + 2)$ flavors. The theory confines in the IR [48]. In the symplectic case there are no baryons and the low energy description consists of a mesonic operator $M = QQ$ with superpotential

$$W = \text{Pf } M \quad (4.1)$$

This theory can be thought as the limiting case of Seiberg duality for an $SP(2N)$ gauge group with $2F$ flavors, where indeed the dual gauge group is $SP(2(F - N - 2))$. For this reason we denote the theory as an $SP(0)$ gauge theory.

It has been shown that in quiver gauge theories, if there are confining symplectic groups, the effective superpotential (4.1) is obtained by wrapping a D instanton on the singularity associated to that node [12, 20]. For example, consider a IIA description of an elliptic quiver, with a circular D4 brane intersecting NS and NS' branes. By adding fractional branes the number of D4 branes between two consecutive and non parallel NS branes is reduced by an HW transition. We can add also orientifolds to this geometry, O4 or O6 planes. The O4 can be put on the circular D4 branes and it switches its charge each time it crosses an NS branes. In this case we have a quiver with alternating SO/SP groups. On the other hand, if we can consider the action of the O6 planes there can be both real and unitary gauge groups.

Consider a node of an elliptic quiver associated to a segment with $2N_i$ D4 branes, stretched between a pair of NS branes. When an $O4^+$ or an $O6^-$ plane⁵ acts on this node the gauge group is projected to $SP(2N_i)$. If $N_i = 0$ the group is $SP(0)$. Nevertheless if we consider a $D(-1)$ brane wrapping this node we have an $O(1)$ instanton. The orientifold projects out the extra fermionic zero modes in the ADHM construction and the instanton contributes to the superpotential. There are only two Ganor strings [26] connecting this nodes to the other(s) and the instanton contributes to the superpotential with a contribution of the form (4.1), where the meson is obtained in terms of the other bi-fundamentals of the quiver.

This construction can be used also for non quiver theories, for example for the $SP(2N)$ SQCD with $2(N + 2)$ flavors discussed above. There are two possible brane configurations. In one case we put an $O6^-$ plane orthogonal to the stack of $N + 2$ D4 branes. The orientifold is extended along the direction $(0, 1, 2, 3, 4, 5)$. The NS and the D6 branes in the setup are in this case rotated along $(4, 5)$ and $(8, 9)$ to preserve $\mathcal{N} = 1$ supersymmetry in

⁴We use the convention $SP(2) \simeq SU(2)$.

⁵The sign represents the action of the orientifold on the NS sector. The charge is associated to the projection of an $SU(2N_i)$ gauge theory.

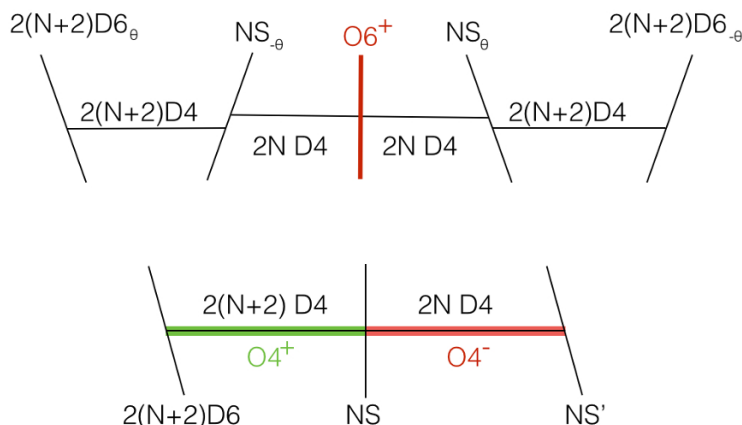


Figure 2. In this figure we reproduce the IIA brane setup describing $SP(2N)$ SQCD with $2(N+2)$ flavors. We represent both the realizations in terms of the $O6$ and $O4$ plane.

four dimensions. In the second case we consider the setup of section 2 and add on a stack of $2(N+2)$ D4 branes an $O4^+$ planes. It becomes an $O4^-$ plane when the NS brane is crossed. The two cases are summarized in figure 2. We can add the D instanton in both cases and perform an HW transition. After the transition the number of D4 branes extended between the NS and the NS' brane vanishes because of charge conservation.⁶ Even in absence of D4 branes the D instanton contributes to the superpotential. The exotic contribution is

$$W = \int d\alpha e^{\alpha M \alpha^T} = \text{Pf } M \quad (4.2)$$

and coincides with (4.1).

Reduction to 3d. Also in this case we can dimensionally reduce the theories to 3d. When the $SP(2N)$ theory is reduced on the circle there is an extra superpotential term, ηY , as explained in [1]. Moreover one can flow to a pure 3d theory by assigning an opposite large real mass to two fundamentals.

In the confined case we use again the prescription of [15]: when considering the theory on the circle we keep the same field content and interactions of the 4d case. This theory can be further reduced to a pure 3d theory. This is done by assigning the real mass to the meson consistently with the masses of the fundamental flavors. In this case there are two massless components in the low energy spectrum: one of them corresponds to the reduced meson $M_{\text{red.}}$ of the theory with $2(N+1)$ flavors while the second massless field is the component M_{2N+3}^{2N+4} . The superpotential of the 3d theory is $W = M_{2N+3}^{2N+4} \text{Pf } M_{\text{red.}}$. The term M_{2N+3}^{2N+4} has the same quantum numbers of the electric monopole Y parametrizing the Coulomb branch of the electric theory with $2(N+1)$ flavors in the pure 3d case. By pursuing this identification the superpotential of the dual theory is

$$W = Y \text{Pf } M_{\text{red.}} \quad (4.3)$$

⁶This charge is the linking number defined in [27]. In this case the cancellation occurs because the orientifold modifies this charge.

Brane interpretation. We can interpret the reduction of the 4d confining $SP(2N)$ theory with $2(N+2)$ flavors and of its confined phase in terms of D branes.

We consider the system discussed above. Here we focus on the case with the O4 planes, a similar discussion applies to the case with the O6 plane. We compactify the direction x_3 and T-dualize along this direction. After T-duality we have a stack of $2N$ D3 branes stretched between an NS and NS' brane.

The D6 branes become D5 branes and the D0 instanton becomes a D1 brane, extended along x_3 . When this brane wraps the compact direction x_3 it encounters two orientifold planes, because after T-duality the $O4^+$ splits into the pair $(O3^+, O3^+)$ [49]. The first plane is fixed at $x_3 = 0$ and the second one is at the *mirror* point $\pi\alpha'/r$ [6]. Here we consider a situation with $2(N+2)$ D5 branes such that the dual theory, obtained by HW transition, becomes an $SP(0)$ theory, i.e. there are no D3 left between the NS branes. We can study the contribution of the D1 branes to the theory or of their magnetic duals, corresponding to the F1 strings. Their contribution to the effective action descends from the contribution of the stringy instanton, corresponding to the superpotential (4.2).

The flow to the pure 3d limit follows from integrating out two fundamentals with large real mass. On the brane side this real mass is obtained by moving two D5 branes at the mirror point on the T-dual circle. The electric theory at $x_3 = 0$ is an $SP(2N)$ theory with $2(N+1)$ fundamentals. At the mirror point the D5s do not give any effect in the HW transition because they cancel against the orientifold charge. In this case there is an ηY superpotential but no extra sectors. In the dual picture moving the two D5s at the mirror point has the same effect discussed in [6]: this effect becomes a scale matching on the mesonic superpotential. The extra D5 branes at the mirror point break the meson into two massless components. This breaking induces the superpotential (4.3). This interaction involves the massless fields and it has to be considered also in the large mass limit.

The partition function. We conclude the analysis by studying the reduction of the confining $SP(2N)$ theory as seen by localization. As in the unitary case we consider the identity between the supersymmetric partition functions on $S^3 \times S^1$ [42] and reduce it to the identity between the partition functions on the squashed three sphere S_b^3 . In the case with $2(N+2)$ fundamentals the relation between the squashed three sphere partition functions of the two phases considered on the circle is

$$\int \prod_{i=1}^N d\sigma_i \left(\prod_{a=1}^{2(N+2)} \Gamma_h(\pm\sigma + \mu_a) \right) \Gamma_h^{-1}(\pm 2\sigma_i) \prod_{1 \leq i < j \leq N} \Gamma_h^{-1}(\pm\sigma_i \pm \sigma_j) = \prod_{1 \leq a < b \leq 2(N+2)} \Gamma_h(\mu_a + \mu_b) \quad (4.4)$$

where we have to enforce the constraint $\sum \mu_a = 2\omega$. This constraint corresponds to the presence of the superpotential ηY in the electric theory and to the superpotential (4.2) in the dual phase. The flow that reduces this duality to a pure 3d one is obtained by assigning the real masses as

$$\mu_a = \begin{cases} m_a + m_A + \omega\Delta_1 & a = 1, \dots, 2N+2 \\ m - (N+1)m_A + \omega(1 - (N+1)\Delta_1) \\ -m - (N+1)m_A + \omega(1 - (N+1)\Delta_1) \end{cases} \quad (4.5)$$

and computing the large m limit. The expected partition function for the $SP(2N)$ theory with $2(N+1)$ fundamental flavors and without the superpotential ηY is obtained in the large m limit. This is computed by using formula (3.12). In the magnetic theory we obtain the contribution of the reduced meson M_{red} and, in addition, we have an extra term of the form

$$\Gamma_h\left(2\omega(1 - (N+1)\Delta_1) - 2(N+1)m_A\right) = \Gamma_h\left(2\omega - \sum \mu_a\right) \quad (4.6)$$

where $\sum \mu_a = 2(N+1)m_A$. This corresponds to the contribution of the electric monopole Y acting as a singlet in the dual phase.

5 Conclusions

In this paper we studied the dimensional reduction of a class of 4d s-confining theories, in the field theory and in the string theory regime, and we obtained 3d dualities. In the string theory regime the structure of the 4d interaction of the IR confined theories is determined by an exotic D instanton configuration. This exotic instanton corresponds, in the T-dual picture, to the effect of a D1 brane, at finite size. We showed that this effect plays a key role in the reduction of the duality in the brane embedding. We checked also the validity of the dualities by studying the reduction of the identities between the supersymmetric partition functions on $S^3 \times S^1$, obtained from localization, to the 3d partition functions on the squashed three sphere S_b^3 . We did not discuss the reduction of orthogonal theories. It would be interesting to perform the analysis in the $SO(N)$ case for both even and odd N . Another interesting aspect regards the reduction of the $\mathcal{N} = 2$ stringy instanton studied in [50–53].

Acknowledgments

We are grateful to Claudius Klare and Alberto Mariotti for comments on the draft. A.A. is funded by the European Research Council (ERC-2012-ADG_20120216) and acknowledges support by ANR grant 13-BS05-0001. A.A. would like to thank CCNY, UCSD, Milano-Bicocca and Bern University for hospitality during various stages of this work.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] O. Aharony, S.S. Razamat, N. Seiberg and B. Willett, *3d dualities from 4d dualities*, *JHEP* **07** (2013) 149 [[arXiv:1305.3924](https://arxiv.org/abs/1305.3924)] [[INSPIRE](#)].
- [2] O. Aharony, S.S. Razamat, N. Seiberg and B. Willett, *3d dualities from 4d dualities for orthogonal groups*, *JHEP* **08** (2013) 099 [[arXiv:1307.0511](https://arxiv.org/abs/1307.0511)] [[INSPIRE](#)].
- [3] V. Niarchos, *Seiberg dualities and the 3d/4d connection*, *JHEP* **07** (2012) 075 [[arXiv:1205.2086](https://arxiv.org/abs/1205.2086)] [[INSPIRE](#)].

- [4] O. Aharony, *IR duality in $d = 3$ $N = 2$ supersymmetric $\text{USp}(2N_c)$ and $\text{U}(N_c)$ gauge theories*, *Phys. Lett. B* **404** (1997) 71 [[hep-th/9703215](#)] [[INSPIRE](#)].
- [5] A. Amariti, D. Forcella, C. Klare, D. Orlando and S. Reffert, *The braneology of 3D dualities*, *J. Phys. A* **48** (2015) 265401 [[arXiv:1501.06571](#)] [[INSPIRE](#)].
- [6] A. Amariti, D. Forcella, C. Klare, D. Orlando and S. Reffert, *4D/3D reduction of dualities: mirrors on the circle*, *JHEP* **10** (2015) 048 [[arXiv:1504.02783](#)] [[INSPIRE](#)].
- [7] N. Seiberg, *Electric-magnetic duality in supersymmetric non-Abelian gauge theories*, *Nucl. Phys. B* **435** (1995) 129 [[hep-th/9411149](#)] [[INSPIRE](#)].
- [8] C. Csáki, M. Schmaltz and W. Skiba, *A systematic approach to confinement in $N = 1$ supersymmetric gauge theories*, *Phys. Rev. Lett.* **78** (1997) 799 [[hep-th/9610139](#)] [[INSPIRE](#)].
- [9] N. Seiberg, *Exact results on the space of vacua of four-dimensional SUSY gauge theories*, *Phys. Rev. D* **49** (1994) 6857 [[hep-th/9402044](#)] [[INSPIRE](#)].
- [10] D. Tong, *TASI lectures on solitons: instantons, monopoles, vortices and kinks*, in *Proceedings of Theoretical Advanced Study Institute in Elementary Particle Physics: Many Dimensions of String Theory (TASI 2005)*, Boulder U.S.A., 5 Jun–1 Jul 2005 [[hep-th/0509216](#)] [[INSPIRE](#)].
- [11] R. Blumenhagen, M. Cvetič, S. Kachru and T. Weigand, *D-brane instantons in type II orientifolds*, *Ann. Rev. Nucl. Part. Sci.* **59** (2009) 269 [[arXiv:0902.3251](#)] [[INSPIRE](#)].
- [12] I. Garcia-Etxebarria and A.M. Uranga, *Non-perturbative superpotentials across lines of marginal stability*, *JHEP* **01** (2008) 033 [[arXiv:0711.1430](#)] [[INSPIRE](#)].
- [13] D. Krefl, *A gauge theory analog of some ‘stringy’ D-instantons*, *Phys. Rev. D* **78** (2008) 066004 [[arXiv:0803.2829](#)] [[INSPIRE](#)].
- [14] A. Amariti, L. Girardello and A. Mariotti, *Stringy instantons as strong dynamics*, *JHEP* **11** (2008) 041 [[arXiv:0809.3432](#)] [[INSPIRE](#)].
- [15] C. Csáki, M. Martone, Y. Shirman, P. Tanedo and J. Terning, *Dynamics of 3D SUSY gauge theories with antisymmetric matter*, *JHEP* **08** (2014) 141 [[arXiv:1406.6684](#)] [[INSPIRE](#)].
- [16] A. Amariti, C. Csáki, M. Martone and N. Rey-Le Lorier, *From S-confinement to 3D chiral theories: dressing the monopoles*, [arXiv:1506.01017](#) [[INSPIRE](#)].
- [17] M. Aganagic, K.A. Intriligator, C. Vafa and N.P. Warner, *The glueball superpotential*, *Adv. Theor. Math. Phys.* **7** (2003) 1045 [[hep-th/0304271](#)] [[INSPIRE](#)].
- [18] I. Garcia-Etxebarria, *D-brane instantons and matrix models*, *JHEP* **07** (2009) 017 [[arXiv:0810.1482](#)] [[INSPIRE](#)].
- [19] B. Florea, S. Kachru, J. McGreevy and N. Saulina, *Stringy instantons and quiver gauge theories*, *JHEP* **05** (2007) 024 [[hep-th/0610003](#)] [[INSPIRE](#)].
- [20] R. Argurio, M. Bertolini, G. Ferretti, A. Lerda and C. Petersson, *Stringy instantons at orbifold singularities*, *JHEP* **06** (2007) 067 [[arXiv:0704.0262](#)] [[INSPIRE](#)].
- [21] O. Aharony and S. Kachru, *Stringy instantons and cascading quivers*, *JHEP* **09** (2007) 060 [[arXiv:0707.3126](#)] [[INSPIRE](#)].
- [22] C. Petersson, *Superpotentials from stringy instantons without orientifolds*, *JHEP* **05** (2008) 078 [[arXiv:0711.1837](#)] [[INSPIRE](#)].
- [23] S. Kachru and D. Simic, *Stringy instantons in IIB brane systems*, [arXiv:0803.2514](#) [[INSPIRE](#)].

- [24] R. Argurio, G. Ferretti and C. Petersson, *Instantons and toric quiver gauge theories*, *JHEP* **07** (2008) 123 [[arXiv:0803.2041](#)] [[INSPIRE](#)].
- [25] G. Ferretti and C. Petersson, *Non-perturbative effects on a fractional D3-brane*, *JHEP* **03** (2009) 040 [[arXiv:0901.1182](#)] [[INSPIRE](#)].
- [26] O.J. Ganor, *A note on zeros of superpotentials in F-theory*, *Nucl. Phys. B* **499** (1997) 55 [[hep-th/9612077](#)] [[INSPIRE](#)].
- [27] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles and three-dimensional gauge dynamics*, *Nucl. Phys. B* **492** (1997) 152 [[hep-th/9611230](#)] [[INSPIRE](#)].
- [28] J.H. Brodie, *Fractional branes, confinement and dynamically generated superpotentials*, *Nucl. Phys. B* **532** (1998) 137 [[hep-th/9803140](#)] [[INSPIRE](#)].
- [29] J. de Boer, K. Hori, Y. Oz and Z. Yin, *Branes and mirror symmetry in $N = 2$ supersymmetric gauge theories in three-dimensions*, *Nucl. Phys. B* **502** (1997) 107 [[hep-th/9702154](#)] [[INSPIRE](#)].
- [30] N.M. Davies, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *Gluino condensate and magnetic monopoles in supersymmetric gluodynamics*, *Nucl. Phys. B* **559** (1999) 123 [[hep-th/9905015](#)] [[INSPIRE](#)].
- [31] N.M. Davies, T.J. Hollowood and V.V. Khoze, *Monopoles, affine algebras and the gluino condensate*, *J. Math. Phys.* **44** (2003) 3640 [[hep-th/0006011](#)] [[INSPIRE](#)].
- [32] O. Aharony, A. Hanany, K.A. Intriligator, N. Seiberg and M.J. Strassler, *Aspects of $N = 2$ supersymmetric gauge theories in three-dimensions*, *Nucl. Phys. B* **499** (1997) 67 [[hep-th/9703110](#)] [[INSPIRE](#)].
- [33] G. Festuccia and N. Seiberg, *Rigid supersymmetric theories in curved superspace*, *JHEP* **06** (2011) 114 [[arXiv:1105.0689](#)] [[INSPIRE](#)].
- [34] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209 [[hep-th/0510251](#)] [[INSPIRE](#)].
- [35] C. Romelsberger, *Counting chiral primaries in $N = 1$, $d = 4$ superconformal field theories*, *Nucl. Phys. B* **747** (2006) 329 [[hep-th/0510060](#)] [[INSPIRE](#)].
- [36] B. Assel et al., *The Casimir energy in curved space and its supersymmetric counterpart*, *JHEP* **07** (2015) 043 [[arXiv:1503.05537](#)] [[INSPIRE](#)].
- [37] F.A.H. Dolan, V.P. Spiridonov and G.S. Vartanov, *From 4d superconformal indices to 3d partition functions*, *Phys. Lett. B* **704** (2011) 234 [[arXiv:1104.1787](#)] [[INSPIRE](#)].
- [38] A. Gadde and W. Yan, *Reducing the 4d index to the S^3 partition function*, *JHEP* **12** (2012) 003 [[arXiv:1104.2592](#)] [[INSPIRE](#)].
- [39] Y. Imamura, *Relation between the 4d superconformal index and the S^3 partition function*, *JHEP* **09** (2011) 133 [[arXiv:1104.4482](#)] [[INSPIRE](#)].
- [40] P. Agarwal, A. Amariti, A. Mariotti and M. Siani, *BPS states and their reductions*, *JHEP* **08** (2013) 011 [[arXiv:1211.2808](#)] [[INSPIRE](#)].
- [41] N. Hama, K. Hosomichi and S. Lee, *SUSY gauge theories on squashed three-spheres*, *JHEP* **05** (2011) 014 [[arXiv:1102.4716](#)] [[INSPIRE](#)].
- [42] V.P. Spiridonov and G.S. Vartanov, *Elliptic hypergeometry of supersymmetric dualities*, *Commun. Math. Phys.* **304** (2011) 797 [[arXiv:0910.5944](#)] [[INSPIRE](#)].

- [43] V.P. Spiridonov and G.S. Vartanov, *Vanishing superconformal indices and the chiral symmetry breaking*, *JHEP* **06** (2014) 062 [[arXiv:1402.2312](#)] [[INSPIRE](#)].
- [44] F. van de Bult, *Hyperbolic hypergeometric functions*, Ph.D. Thesis, University of Amsterdam (2008), <http://www.its.caltech.edu/~vdbult/Thesis.pdf>.
- [45] D.L. Jafferis, *The exact superconformal R-symmetry extremizes Z*, *JHEP* **05** (2012) 159 [[arXiv:1012.3210](#)] [[INSPIRE](#)].
- [46] N. Hama, K. Hosomichi and S. Lee, *Notes on SUSY gauge theories on three-sphere*, *JHEP* **03** (2011) 127 [[arXiv:1012.3512](#)] [[INSPIRE](#)].
- [47] F. Benini, C. Closset and S. Cremonesi, *Comments on 3d Seiberg-like dualities*, *JHEP* **10** (2011) 075 [[arXiv:1108.5373](#)] [[INSPIRE](#)].
- [48] K.A. Intriligator and P. Pouliot, *Exact superpotentials, quantum vacua and duality in supersymmetric $SP(N_c)$ gauge theories*, *Phys. Lett. B* **353** (1995) 471 [[hep-th/9505006](#)] [[INSPIRE](#)].
- [49] A. Hanany and B. Kol, *On orientifolds, discrete torsion, branes and M-theory*, *JHEP* **06** (2000) 013 [[hep-th/0003025](#)] [[INSPIRE](#)].
- [50] H. Ghorbani, D. Musso and A. Lerda, *Stringy instanton effects in $N = 2$ gauge theories*, *JHEP* **03** (2011) 052 [[arXiv:1012.1122](#)] [[INSPIRE](#)].
- [51] H. Ghorbani and D. Musso, *Stringy instantons in $SU(N)$ $N = 2$ non-conformal gauge theories*, *JHEP* **12** (2011) 070 [[arXiv:1111.0842](#)] [[INSPIRE](#)].
- [52] R. Argurio, D. Forcella, A. Mariotti, D. Musso and C. Petersson, *Field theory interpretation of $N = 2$ stringy instantons*, *JHEP* **02** (2013) 002 [[arXiv:1211.1884](#)] [[INSPIRE](#)].
- [53] H. Ghorbani, *Gauge-stringy instantons in $\mathcal{N} = 2$ $U(N)$ gauge theories*, *JHEP* **12** (2013) 041 [[arXiv:1306.1487](#)] [[INSPIRE](#)].